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by

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Abstract

The numerical Schwarz algorithm, [1], for solving elliptic partial differential equations is essentially a block Gauss-Siedel method for inverting a matrix equation. The numerical Schwarz algorithm is only one variant of the method of overlapping domains. This method yields matrix equations that are related to the standard systems obtained from elliptic P.D.E.s., [2]. This paper analyzes the use of Jacobi splittings on these matrix equations.

1. The Linear Systems

We consider the solution of the linear system

(1.1) $\quad Tx = b$

where T is a pxp block matrix of the form

(1.2)
$$T = \begin{bmatrix} F_1 & G_1 & & \\ E_1 & F_2 & G_2 & \\ & \ddots & \ddots & G_{p-1} \\ & & E_p & F_p \end{bmatrix}$$

where

(1.3)
$$F_1 = \begin{bmatrix} T_1 & R_1 \\ R^t_1 & T_2 \end{bmatrix}, \quad F_p = \begin{bmatrix} T_{2p-2} & R_{2p-2} \\ R_{2p-2} & T_{2p-1} \end{bmatrix}$$

$$F_i = \begin{bmatrix} T_{2i} & R_{2i} & \\ R_{2i} & T_{2i+1} & R_{2i+1} \\ & R^t_{2i+1} & T_{2i+2} \end{bmatrix}, \quad 2 \le i \le p-1$$

$$G_i = \begin{bmatrix} 0 & 0 \\ 0 & R_{2i} \end{bmatrix}, \quad 1 < i < p - 1$$

$$E_i = \begin{bmatrix} R^t_{2i-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad 2 \le i \le p$$

Each of the matrices R_i and T_i are $n_i \times n_i$ where $0 < n_i < n$, n a given integer, and further, each T_i is symmetric and non-singular. The source vector $b = [b_1, b_2, .., b_p]^t$ where

$$b_1 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad b_p = \begin{bmatrix} \beta_{2p-2} \\ \beta_{2p-1} \end{bmatrix},$$

(1.5)
$$b_i = \begin{bmatrix} \beta_{2i} \\ \beta_{2i+1} \\ \beta_{2i+2} \end{bmatrix}, \quad 1 \le i \le p - 2 \;,$$

$$\beta_i = n_i \times 1 \text{ column vector} .$$

Systems (1.1) arise from the numerical solution of elliptic partial differential equations by the method of overlapping domains, cf. [1].

Associated with system (1.1) is

(1.6) $\quad T's' = b'$

where

(1.7)
$$T' = \begin{bmatrix} T_1 & R_1 & & \\ R^t_1 & T_2 & R_2 & \\ & \ddots & \ddots & R_{2p-2} \\ & & R^t_{2p-2} & T_{2p-1} \end{bmatrix}$$

(1.8) $\quad b' = [\beta_1, \beta_2, \ldots, \beta_{2p-1}]^t.$

In [2], it was shown that if

$Ts = b$ where $s = [s_1, s_2 .., s_p]^t$ and

$$s_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad s_p = \begin{bmatrix} x_{3p} \\ x_{3p+1} \end{bmatrix},$$

(1.9)

$$s_i = \begin{bmatrix} x_{3i} \\ x_{3i+1} \\ x_{3i+2} \end{bmatrix}, \quad 2 \le i \le p-1,$$

then

$$s' = [x_1, x_3, x_4, x_6, \ldots, x_{3i}, x_{3i+1}, \ldots, x_{3p}, x_{3p+1}]^t$$

is a solution of $T's' = b'$.

Hence, a solution of (1.6) can be constructed from a solution of (1.1). We consider first order iterative methods for the solution of (1.1) and compare their convergence rates to the corresponding iterative method for the solution of (1.6). In particular, Jacobi splittings will be studied because of their usefulness on parallel computers such as a multiprocessor.

The following theorem compares the eigenvalue of T with those of T'. We adopt the notation $\lambda(Q)$ to denote the set of eigenvalues of a matrix Q.

Theorem 1.1

$$\lambda(T) \subseteq \lambda(T') \cup [\bigcup_{i=1}^{p-1} \lambda(T_{2i})].$$

Proof:

Suppose $TX = \mu X$ where $S = [X_1, X_2, \ldots, X_p]^t$.

Partioning X_i, $1 \le i \le p$, according to (1.3) we have

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad X_p = \begin{bmatrix} x_{3p} \\ x_{3p+1} \end{bmatrix},$$

$$X_i = \begin{bmatrix} x_{3i} \\ x_{3i+1} \\ x_{3i+2} \end{bmatrix}, \quad 2 \le i \le p-1.$$

Hence,

$$(T_{2i} - \mu I)x_{3i+2} = (T_{2i} - \mu I)x_{3i+3}.$$

If $\mu \epsilon \lambda(T_{2i})$,

then $x_{3i+2} = x_{3i+3}$ and

$$X' = [x_1, x_3, x_4, x_6, \ldots, x_{3i}, x_{3i+1}, \ldots x_{3p+1}]^t$$

is an eigenvector of T' corresponding to eigenvalue μ.

2. Jacobi Methods

Consider a Jacobi splitting of (1.6), that is, $T' = M' - N'$ where

(2.1)
$$M' - N' = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_{2p-1} \end{bmatrix} - \begin{bmatrix} E_1 & G_1 & & & \\ G_1^t & E_2 & G_2 & & \\ & G_2^t & \ddots & \ddots & \\ & & \ddots & \ddots & G_{2p-2} \\ & & & G_{2p-2}^t & E_{2p-1} \end{bmatrix}$$

This gives rise to a corresponding Jacobi splitting $T = M - N$ in (1.1),

(2.2)
$$M-N = \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \\ & & & \Delta_p \end{bmatrix} - \begin{bmatrix} H_1 & J_1 & & \\ K_2 & H_2 & \ddots & \\ & \ddots & \ddots & J_{p-1} \\ & & K_p & H_p \end{bmatrix}$$

where

$$\Delta_1 = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}, \quad \Delta_p = \begin{bmatrix} D_{2p-2} & 0 \\ 0 & D_{2p-1} \end{bmatrix},$$

(2.3)
$$\Delta_i = \begin{bmatrix} D_{2i-2} & & \\ & D_{2i-1} & \\ & & D_{2i} \end{bmatrix}, \quad 2 \le i \le p-1,$$

$$H_1 = \begin{bmatrix} E_1 & G_1 \\ G_1^t & E_2 \end{bmatrix}, \quad H_p = \begin{bmatrix} E_{2p-2} & G_{2p-2} \\ G_{2p-2}^t & E_{2p-1} \end{bmatrix} ,$$

$$(2.4) \qquad H_{i-1} = \begin{bmatrix} E_{2i} & G_{2i} & \\ G_{2i}^t & E_{2i+1} & G_{2i+1} \\ & G_{2i+1}^t & E_{2i+2} \end{bmatrix}, \quad 2 \le i \le p-1,$$

$$(2.6) \qquad J_i = \begin{bmatrix} 0 & 0 \\ 0 & G_{2i} \end{bmatrix}, \quad 1 \le i \le p-1,$$

$$(2.7) \qquad K_i = \begin{bmatrix} G_{2i-1}^t & 0 \\ 0 & 0 \end{bmatrix}, \quad 2 \le i \le p .$$

We have the following general result relating the eigenvalues of $[M']^{-1} N'$ to those $M^{-1} N$.

<u>Theorem 2.1</u> If D_{2i}, $1 \le i \le p$, are nonsingular, then

$$\lambda[(M')^{-1}N'] \subset \lambda(M^{-1}N) \cup [\bigcup_{i=1}^{p} \lambda(D_{2i}^{-1} E_{2i})] .$$

<u>Proof</u>:

Suppose $MX = \lambda NX$ for some eigenvalue λ. If $X = [X_1, X_2, \ldots, X_p]^t$, then for $i = 1, \ldots, p$

$$A_i X_i = \lambda(K_i X_{i-1} + H_i X_i + J_i X_{i+1})$$

where

$$K_1 = J_p = 0.$$

Partitioning X_i, $1 \le i \le p$, according to (1.3) we have

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad X_p = \begin{bmatrix} x_{3p} \\ x_{3p+1} \end{bmatrix},$$

$$X_i = \begin{bmatrix} x_{3i} \\ x_{3i+1} \\ x_{3i+2} \end{bmatrix}, \quad 2 \le i \le p-1 .$$

Hence,

$$D_{2i} x_{3i+2} = \lambda (G_{2i+1}^t x_{3i+1} + E_{2i} x_{3i+1} + G_{2i} x_{3i+4})$$

$$D_{2i} x_{3i+3} = \lambda (G_{2i+1}^t x_{3i+1} + E_{2i} x_{3i+3} + G_{2i} x_{3i+4})$$

or

$$(D_{2i} - \lambda E_{2i+1})\, x_{3i+2} = (D_{2i} - \lambda E_{2i})\, x_{3i+3}.$$

If $(D_{2i} - \lambda E_{2i})$ is nonsingular for $1 \le i \le p-1$, then $x_{3i+2} = x_{3i+3}$, $1 \le i \le p-1$, so that

$$X' = [x_1, x_3, x_4, x_6, \ldots, x_{3i}, {}_{3i+1}, \ldots, x_{3p}, x_{3p+1}]^t$$

is an eigenvector of $(M')^{-1}N'$ corresponding to the eigenvalue λ.

<u>Corollary</u> If $E_{2i} = 0$, $1 \le p$, and D_{2i}, $1 \le i \le p$, are nonsingular, then

$$\lambda[(M')^{-1}N'] = \lambda(M^{-1}N) .$$

3. Point-Jacobi Splitting for Laplace's Equation

Consider the system (1.6) where

$$(3.1) \qquad T = \frac{1}{h^2} \begin{bmatrix} F_1 & G_1 \\ G_2 & F_2 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} T_1 & R_1 \\ R_1^t & T_2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} T_2 & R_1 \\ R_1^t & T_3 \end{bmatrix},$$

$$(3.2) \qquad T_1 = T_3 = \left.\begin{bmatrix} A & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & A \end{bmatrix}\right\} r,$$

$$T_2 = \left.\begin{bmatrix} A & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & A \end{bmatrix}\right\} k ,$$

$$R_1 = \left.\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & & \vdots \\ -I & 0 \cdots & 0 \end{bmatrix}\right\} r,$$

$$A = \left.\begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}\right\} n .$$

Then, T' is matrix obtained from a 5-point discretization with mesh size $h = 1/n+1$ of Laplace's equation on the unit square. Consider $T = M - N$ where $M = (4/h^2)I$. Then, $N = M - T$ so that $M^{-1}N = I - (h^2/4)T$. Also, in (2.3),

$D_2^{-1}E_2 = I - (h^2/r)T_2$. Hence by Theorem 2.1,

$$\lambda[(M')^{-1}N'] = \{1 - (h^2/4)\ \lambda(T')\}$$

$$\subseteq \lambda(M^{-1}N)$$

$$= \lambda(I - (h^2/4)T)$$

$$= \{1 - (h^2/4)\ \lambda(T)\} \tag{3.3}$$

$$\subseteq \lambda[(M')^{-1}N'] \cup \lambda(D_2^{-1}E_2)$$

$$= \{1 - (h^2/4)\ \lambda(T')\} \cup \{1 - (h^2/4)\ \lambda(T_2)\}\ .$$

Theorem 3.1

For the system given by (3.1),

$$\rho[(M')^{-1}N'] = \rho(M^{-1}N)\ .$$

Proof

By Theorem 1.1, $\lambda(T') \subseteq \lambda(T) \subseteq \lambda(T') \cup \lambda(T_2)$

so that the eigenvalues of T are real. We show that

$$\min_{\upsilon \in \lambda(T')} \{\upsilon\} < \min_{\mu \in \lambda(T_2)} \{\mu\}$$

and

$$\max_{\upsilon \in \lambda(T')} \{\upsilon\} > \max_{\mu \in \lambda(T_2)} \{\mu\}\ .$$

The result will then follow from (3.3)

Note that $\mu \in \lambda(T_2)$ implies

$$\mu = 1/h^2[4 - 2\cos(p\pi/n) - 2\cos(q\pi/n)]$$

$$\text{for } p = 1, 2, \ldots, k-1$$

$$q = 1, 2, \ldots, n-1\ .$$

Hence,

$$\min_{\mu \in \lambda(T_2)} \{\mu\} = (1/h^2)\,[4 - 2\cos(\pi/k) - 2\cos(\pi/n)]\ ,$$

$$\max_{\mu \in \lambda(T_2)} \{\mu\} = (1/h)[4 + 2\cos(\pi/k) + 2\cos(\pi/n)]$$

Now, $\upsilon \in \lambda(T')$ implies

$$\upsilon = (1/h^2)\,[4 - 2\cos(p\pi/n) - 2\cos(q\pi/n)$$

$$\text{for } p, q = 1, \ldots, n-1\ .$$

Hence

$$\min_{\upsilon \in \lambda(T')} \{\upsilon\} = (1/h^2)\,\{4 - 4\cos(\pi/n)\}$$

and

$$\max_{\upsilon \in \lambda(T')} \{\upsilon\} = (1/h^2)\,\{4 + 4\cos(\pi/n)\}\ .$$

4. Computational Results

We solve the system (3.1) with the splitting (2.2) where

$$D_i = L_i L_i^t \quad , i = 1,2\ ,$$

is the incomplete Cholesky factorization of R_i and F_2, respectively, in (3.1). We vary the integer k in (3.2) to determine if the iteration count is invariant to k thus, a theorem such as Theorem 3.1 might hold. Table 1 records the results with $\Delta x = 1/31$ and $\Delta y = 1/65$ for Laplace's equation $\Delta u = 0$ and $u = 2$ on boundary

Table 1

k	iterations
0	648
5	644
15	642
25	642
30	642
35	642

References

[1] Miller, K., "Numerical Analogs to the Schwarz Alternating Procedure", Numer. Math., 7 (1965), pp 91-103

[2] Rodrigue, G., and Simon, J., "A Generalization of the Numerical Schwarz Algorithm", Proceedings of 6th Int. Conf. on Comp. Meth. in App. Sci. and Eng., INRIA, Dec. 1983.

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